Exercise 31

Prove the statement using the ε, δ definition of a limit.

$$\lim_{x \to -2} (x^2 - 1) = 3$$

Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad 0 < |x - (-2)| < \delta \qquad \text{then} \qquad |(x^2 - 1) - 3| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x + 2|$.

$$|(x^2 - 1) - 3| < \varepsilon$$

$$|x^2 - 4| < \varepsilon$$

$$|(x - 2)(x + 2)| < \varepsilon$$

$$|x - 2||x + 2| < \varepsilon$$

On an interval centered at $x = -2$, a positive constant C can be chosen so that $|x - 2| < C$.

$$C|x + 2| < \varepsilon$$

$$|x + 2| < \frac{\varepsilon}{C}$$

To determine C, suppose that x is within a distance a from -2.

$$|x + 2| < a$$

$$-a < x + 2 < a$$

$$-a - 4 < x - 2 < a - 4$$

$$|x - 2| < |-a - 4| = a + 4$$

The constant C is then $a + 4$. Choose δ to be whichever is smaller between a and $\varepsilon/(a + 4)$: $\delta = \min\{a, \varepsilon/(a + 4)\}$. Now, assuming that $|x + 2| < \delta$,

$$\begin{aligned} |(x^2 - 1) - 3| &= |x^2 - 4| \\ &= |(x - 2)(x + 2)| \\ &= |x - 2||x + 2| \\ &< (a + 4)\left(\frac{\varepsilon}{a + 4}\right) = \varepsilon. \end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x \to -2} (x^2 - 1) = 3.$$